Research plan: Multipartite Entanglement, Monogamy Relations, and Quantum Error-Correcting Codes

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1. SUMMARY OF THE RESEARCH PLAN

Quantum states show many intriguing non-classical features, foremost the one of entanglement. These are a type of correlations that cannot be explained by any classical means [1]. Importantly, quantum entanglement is recognized to be a key ingredient for emerging applications that are based on properties of quantum states, such as novel types of computers, secure communication methods, and precision measurements [2]. It also represents a core aspect of many phenomena in solid-state physics, quantum correlations being integral for the collective properties of particles on the mesoscopic scale.

While bipartite entanglement of pure states is well understood, the investigation of entanglement between many particles is a timely and relevant endeavor [3]. In order to recognize the full potential that quantum correlations can provide, it is important to obtain further insights into how entanglement can be shared between many particles. Likewise, error-correcting methods used for quantum information processing strongly rely on features of multipartite entanglement [4]: by distributing information onto many particles, it is possible to recover the information in the case of environmental disturbance or particle loss.

Here, we propose to study entanglement in systems consisting of multiple particles with so-called monogamy relations [5]. These constrain the shareability of quantum correlations, and often involve invariant quantities that are independent of particle exchange and of the local basis chosen. However, only few monogamy relations are known, the most famous one being the Coffman-Kundu-Wootters inequality [6]. An example of such a relation in the context of quantum error-correcting codes is the so-called shadow enumerator, which was introduced by Rains. With it, he obtained some of the strongest bounds on the existence of quantum codes to date [7]. While its physical interpretation was originally unclear, we could show that this shadow enumerator in fact represents monogamy relations of order two, applicable to all finite dimensions [A].

In this project, we aim to derive families of higher-order monogamy relations in a systematic way by means of the generalized shadow inequalities [8]. Rains provided a recipe to obtain these, but so far only one example of order two is known. Our goal is to obtain explicit families of higher-order inequalities, which are monogamy-like relations. This will be done by both analytical as well as computational methods, which additionally helps to mitigate inherent risks of this ambitious project. After interpreting the relations obtained in terms of physically meaningful quantities, we will cast them into corresponding weight enumerators, which are tools for the analysis of quantum error-correcting codes.

Furthermore, these relations have applications for entanglement detection, and compatibility conditions for

reductions to originate from a joint state can be derived [9]. The development of these methods will likely help to settle long-standing questions such as the existence of certain code spaces and of highly entangled states. By establishing new connections between the fields of quantum error-correcting codes and multipartite entanglement, we will also bring these communities closer together. Thus the successful conclusion of this project will not only shed a new light on the nature of quantum correlations shared between multiple parties and on the constraints that govern their distribution, but will also have a broader impact in related fields such as in solid-state physics, theoretical chemistry, and in the theory of classical error-correcting codes.

2. RESEARCH PLAN

2.1. Current state of research in the field

Multipartite entanglement & monogamy relations: A pure quantum state $|\psi_{AB}\rangle$ of two particles is called entangled, if it cannot be written as the tensor product of the single-particle states,

$$|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \,. \tag{1}$$

Thus, the defining feature of pure entangled states is that they are not completely characterized by their single-particle reductions. The paradigmatic example of an entangled state on two subsystems having two levels each is the famous Bell state, $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, which exhibits strong non-classical correlations between the local subsystems.

While the entanglement properties of bipartite pure states are well understood, multipartite states allow for more freedom in the distribution of quantum correlations between the different parties. This gives rise to different classifications of multipartite entanglement with respect to different entanglement properties or operational procedures. As an example, considering local operations and classical communications gives in the case of three qubits rise to two entanglement classes, which are represented by the W- and the Greenberger-Horne-Zeilinger states,

$$\begin{aligned} |W\rangle \quad &= (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}\,, \\ |GHZ_3\rangle &= (|000\rangle + |111\rangle)/\sqrt{2}\,. \end{aligned} \tag{2}$$

However, entanglement cannot be shared arbitrarily among the individual subsystems. Rather, so-called monogamy of entanglement relations constrain the possible correlations which quantum states can exhibit [5]. In its most basic form, this concept can be expressed as follows: If two parties A and B are maximally entangled with each other, then neither A nor B can also be entangled with a third party C. Monogamy relations are different formulations of this concept, imposing restrictions on how quantum correlations can be shared by multiple parties. This was first made precise for 2-level systems in a seminal article by Coffman, Kundu, and Wootters [6],

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq \mathcal{C}_{A(BC)}^2 \,. \tag{3}$$

Above, $\mathcal{C}_{AB}$, $\mathcal{C}_{AC}$, and $\mathcal{C}_{A(BC)}$ denote the squared concurrences between parties A and B, between A and C, and between A and parties BC grouped together, measuring entanglement between these. Considering the three-partite states introduced in Eq. (2), the W-state reaches equality in above equation. In contrast, the GHZ state does not reach equality; its so-called tangle is non-zero and the state contains essential three-way entanglement. A conjecture extending this Coffman-Kundu-Wooters inequality [Eq.(3)] to more parties was later proven by Osborne and Verstraete [10], and similar relations constraining multipartite quantum correlations have been found for other measures such as squashed entanglement, entanglement negativity, and non-local correlations [11–14]. However, systematic methods to derive new monogamy relations are lacking, and novel approaches to the monogamy of entanglement and other correlations would be desirable.

Given a measure of bipartite entanglement, one could argue that those states, which show maximal entanglement across all bipartitions, are the most entangled ones [15]. Such pure states, whose marginals, obtained after tracing out at least half of the parties, are all maximally mixed, are also called absolutely maximally entangled (AME) [16, 17]. For qubits, such states do only exist for $n = 2, 3, 5, 6$ parties [4, 18], [E]. This can be seen as a manifestation of monogamy in quantum states; correlations cannot be distributed arbitrarily among the subsystems, but are constrained. Currently, the existence of AME states of larger dimensions is still an open problem [4]. It is likely that monogamy relations of higher order would help to resolve these and related questions in multipartite entanglement, and will further the understanding on what kind of correlations can arise from quantum states.

Quantum error-correcting codes: In order to perform quantum information processing in the presence of disturbance from the environment, some sort of error correction has to be performed on the information carriers, which are multipartite quantum states. Quantum error-correcting codes (QECC) allow for such a mechanism. More precisely, QECC are subspaces of the Hilbert space that can be reconstructed completely, if a disturbance occurs on few particles only. However, finding good codes is a challenging endeavor, and various techniques have been devised to keep the information imprinted onto the quantum states safe and their manipulation feasible.

Developed over two decades ago in a series of papers by Shor, Laflamme, and Rains [7, 19, 20] and originating in invariant theory [21], quantum weight enumerators are mathematical tools to characterize QECC. Weight enumerators are polynomials in specific expectation values, that are obtained from certain subsets of particles only. These are invariant under the local basis chosen, and crucially, they can be used to exclude the existence of certain codes with the help of linear programming. Aside from the primary and dual weight enumerators from Shor and Laflamme, Rains' shadow enumerator is of notable importance [7]. It is derived from the so-called shadow inequality of order two: For two positive semi-definite Hermitian operators M_1, M_2, and a fixed subset $T \subseteq \{1, \ldots, n\}$, the following inequality holds.

$$\sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S \cap T|} \operatorname{Tr}[\operatorname{Tr}_{S^c}(M_1) \operatorname{Tr}_{S^c}(M_2)] \geq 0 \,, \tag{4}$$

where S^c denotes the complement of S in $\{1, \ldots, n\}$. This resulted in some of the strongest general bounds on the existence of quantum and classical error-correcting codes to date [7, 22].

The existence of certain QECC is related to the existence of certain quantum states, pure states being one-dimensional codes [4]. In this context, Eq. (4) relates reductions of a joint state, constraining the distribution of correlations arising in multipartite quantum systems. Therefore, Eq. (4) is a monogamy relation, and in fact can be used to prove the non-existence of certain putative quantum states, e.g. of higher-dimensional AME states [A].

However, no tight bounds are known for the existence of QECC. This is to be expected, as the main tool at hand, the theory of quantum weight enumerators, does not describe a code completely: The presently used weight enumerators are polynomial invariants of degree two only [21]. While the theory of higher-order invariants has seen many developments [23–30], its application to QECC and multipartite entanglement has been limited. Thus an extension of the quantum weight enumerator theory incorporating these developments in combination with higher-order monogamy relations would be of significant interest. In particular, this could settle long-standing open questions on parameters of putative codes and highly entangled states.

2.2. Current state of own research

States determined by their marginals: The quantum marginal problem asks, whether, given a putative collection of marginals, a joint state either exists or must be unique. We approached this question from a result obtained by Linden et al.: Almost all pure states of three qubits are completely determined by their 2-party reduced density matrices [31]. We showed that a similar statement is not valid for states of five or more qubits. For this, we constructed a witness based on pure states having maximally mixed marginals, and formulated the question as a semi-definite program [B]. We could show in a follow-up article that states of four parties having equal but arbitrary dimensions are indeed also determined by their 2-body reductions, if one considers pure states only [C].

With Simone Severini, I approached this topic from another point of view, extending the famous Ulam graph reconstruction problem to quantum states [D]: Namely the question, if, given a complete but unlabeled collection of marginals of size $(n-1)$, the joint state on n parties may uniquely be determined. Interestingly, the existence of a joint state can in some cases already be excluded when only having access to all unlabeled marginals of size $\lfloor \frac{n}{2} \rfloor + 1$.

In these projects I learned different tools for reasoning about relations that connect joint states to their reductions, which is a core aspect of monogamy relations.

Absolutely maximally entangled states: Here, we resolved a long-standing question in multipartite entanglement, originally posed in the seminal work of Calderbank et al. on quantum error-correcting codes [32]: whether or not a pure state of seven qubits exists, whose three-body marginals are all maximally mixed. Numerical approaches by various researchers suggested the absence of such a state; an analytical proof however remained elusive [4]. By combining properties of such states, obtained from their Schmidt decomposition and anticommutation relations of Pauli operators, we characterized this problem in the Bloch representation [33]. This enabled us not only to obtain a proof for the non-existence of a seven qubit AME state, but in fact could exclude all but the known cases of $n = 2, 3, 5, 6$ qubits, all of which are stabilizer or graph states [E].

During this work I learned to work with the Bloch representation, which will be crucial to analyze the generalized shadow inequalities.

Quantum weight enumerators: As my most recent research line, I have been working on the quantum weight enumerator theory of quantum error-correcting codes. In particular, we obtained a proof of the MacWilliams identity in the Bloch representation, and applied tools from QECC to analyze multipartite entanglement. This resulted in new bounds on the existence of AME states having higher local dimensions [A]. We could show that the main method used for this, the so-called shadow enumerator, represents a family of monogamy relations for the correlations appearing in quantum states. This generalizes relations of order two for qubit-systems that were found by Ref. [13] to all finite dimensions. Furthermore, we showed that a positive but not completely positive map can be derived from it [F], extending the universal state inversion and the reduction criterion for entanglement detection [9, 34]. This interpretation of the shadow enumerator as monogamy relation and its implications are novel, and strengthens the understanding of the connections between QECC and multipartite entangled systems.

The expertise I acquired in this project will be relevant to incorporate the monogamy-like higher-order relations into the quantum weight enumerator machinery.

Project-related publications:

[A] **FH**, C. Eltschka, J. Siewert, and O. Gühne,
Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity.
arXiv:1708.06298 (2017).

[B] **FH** and O. Gühne,
Characterizing Ground and Thermal States of Few-Body Hamiltonians.
Phys. Rev. Lett. **117**, 010403 (2016).

[C] N. Wyderka, **FH**, and O. Gühne,
Almost all four-particle pure states are determined by their two-body marginals.
Phys. Rev. A **96**, 010102(R) (2017).

[D] **FH** and S. Severini,
Some Ulam's reconstruction problems for quantum states.
in preparation.

[E] **FH**, O. Gühne, and J. Siewert,
Absolutely Maximally Entangled States of Seven Qubits Do Not Exist.
Phys. Rev. Lett. **118**, 200502 (2017).

[F] C. Eltschka, **FH**, O. Gühne, and J. Siewert,
Family of Correlation Equalities and Monogamy Relations for Entanglement,
in preparation.

2.3. Detailed research plan

The proposed research aims at answering the following questions:
(1) Which known relations that constrain quantum correlations originate from the generalized shadow inequalities? Can one find new monogamy-like relations for small systems composed of three to four parties, and obtain conditions for the compatibility of reduced states to stem from a joint quantum state?
(2) Can an explicit family of higher-order shadow inequalities be found? Can these be understood as physically meaningful higher-order monogamy relations, and be incorporated into the quantum weight enumerator machinery to characterize quantum error-correcting codes?

The answers to these questions will have strong impacts in both the fields of multipartite entanglement and of quantum error-correcting codes. In particular, we expect to be able to provide stronger bounds on the existence of quantum error-correcting codes and to introduce novel tools for the analysis of entanglement in multipartite states, crucial for the understanding of collective particle behavior in many-body systems on the quantum scale.

Starting with small systems, we aim to both understand existing monogamy relations and invariants in the context of the generalized shadow inequalities, as well as to derive novel types thereof. Then, we intend to obtain an explicit family of higher-order shadow inequalities for an arbitrary number of parties, from which general monogamy-like relations and possibly new types of quantum weight enumerators follow.

A cornerstone of this project is a relation derived by Rains in a paper on polynomial invariants [8] that generalizes Eq. 4, the *generalized shadow inequalities*: For positive semi-definite operators $M_1, \dots, M_k$, and any Hermitian idempotent λ in the group algebra of S_k^n, the n-fold direct product of the permutation group S_k, the following expression is non-negative.

$$\sum_{\pi \in S_k^n} \lambda(\pi) A'_\pi(M_1, M_2 \dots, M_k) \geq 0 \,. \tag{5}$$

In above equation, $A'_\pi(M_1, M_2, \dots, M_k)$ are polynomial invariants, obtained by tracing over permuted subsystems of the operators $M_1, \dots, M_n$. Thus if $M_1 = \dots = M_k$, the generalized shadow inequalities represent relations between polynomial invariants of quantum states or QECC. In the case of $k = 2$, Rains gave an example of such a shadow inequality having order two, namely Eq.(4), leading to the shadow enumerator and a family of monogamy relations [F]. However, aside from a degree four relation presented by Ref. [13], applicable only to two-level systems, no other explicit relations involving higher-order invariants are known. Thus we propose a general approach to such relations, which derives from the generalized shadow inequalities. This will result in explicit new higher-order relations between invariants, and likely will clarify their interpretation as monogamy relations constraining the distribution of correlations.

We are convinced that this program is both feasible as well as of significant interest: higher-oder monogamy relation have already been discovered [13], and the shadow inequality of degree two are indeed monogamy relations with interesting applications [E, F]. From these preliminary results and insights, as well as by judging from the structure of the generalized shadow inequalities [Eq (5)], we are confident that explicit higher-order inequalities will have equally interesting features and applications as those which were found in the case of $k = 2$.

(1) First research line: Higher-order shadow inequalities for a small number of parties.
Here, we focus on systems consisting of a small number of parties. This allows us to analyze existing monogamy relations and invariants [6, 10–14, 24–28, 35, 36] in the context of the generalized shadow inequalities, as well as to analytically derive new higher-order relations. Crucially, Eq. (5) gives a recipe on how to obtain these: A Hermitian idempotent has to be found in the group algebra of S_k^n, the n-fold direct product of the permutation group S_k. For this, we will start with a small number of parties n and a small order k. In light of the results in Ref. [13] concerning order four we are confident that this can be done analytically. Additionally, Rains has shown certain higher-order invariants reduce to those of lower order, which likely will simplify the analysis further [8].

We expect that the relations obtained will also lead to new compatibility conditions between subsystems of a joint state, similar to those found by Butterley et al. [37]: In order that reductions ρ_{AB}, ρ_{AC}, and ρ_{BC} stem from a joint state ρ_{ABC} on three qubits, one has following necessary condition,

$$0 \leq \mathbb{1} - \rho_A - \rho_B - \rho_C + \rho_{AB} + \rho_{AC} + \rho_{BC} \leq \mathbb{1} \,. \tag{6}$$

In fact, this relation can be seen as a direct consequence of the order-two shadow inequality [F], and we aim to obtain similar compatibility conditions from higher-order inequalities. Finally, these will also yield positive but not completely positive maps, which are of potential interest for entanglement detection.

Milestones for the first research line:
M 1.1 Analyze the higher-order shadow inequalities in the context of known monogamy relations and invariants.
M 1.2 Derive explicit higher-order relations for three to four parties.
M 1.3 Obtain compatibility conditions on subsystems to originate from a joint state, and positive maps for entanglement detection.

(2) Second research line: A family of higher-order shadow inequalities.
In this second research line, we intend to construct explicit higher-order shadow inequalities in a systematic way. First, we aim at providing an efficient algorithm to find Hermitian idempotents in the group algebra of S_k^n. With the help of a computer algebra system such as Mathematica or Maple, we will create tables of such idempotents. Both the algorithm as well as the tables will be uploaded to an online repository such as github for the use of other researchers.

Having access to explicit idempotents, one can obtain higher-order shadow inequalities directly from Eq.(5). With these examples at hand, and the insight gained from Milestones 1.1 and 1.2, our goal is to obtain a complete family of monogamy-like relations for invariants. It will be crucial to understand them in terms of physically relevant quantities, which are likely similar to those which were obtained for the inequality of order two.

Finally, we aim to cast the resulting relations into quantum weight enumerators of higher order: Already proposed by Rains [8], this will likely be a powerful method to characterize QECC. As in the case of the degree-two enumerators, it might be possible to formulate linear or quadratic programs to determine higher order weight enumerators of putative codes, and to obtain bounds on possible code parameters. Equipped with these novel tools, we propose to attack long-standing open questions in the field of QECC, such as the existence of a putative stabilizer code having parameters $[[24, 0, 10]]_2$ or the existence of an AME state of four six-level systems [38]. Furthermore, it would be interesting to obtain general bounds on the existence of codes, and in particular to prove a conjecture on so-called quantum maximum distance separable (QMDS) codes: For most QMDS codes, the minimum distance is smaller than the local dimension [39].

Milestones for the second research line:
M 2.1: Provide an efficient algorithm to find Hermitian idempotents, and provide online-tables of those found.
M 2.2: Provide a family of monogamy-like relations in terms of physically relevant quantities.
M 2.3: Construct corresponding weight enumerators. Obtain bounds on QECC and AME states.

Strategies and risks:
While the proposed research springs from the work done during my thesis, namely understanding and applying tools from QECC to problems in multipartite entanglement, the approach we put forward to derive higher-order monogamy relations is to our knowledge completely novel. It represents a significant step ahead from the work done during my PhD. We should also mention that the project is ambitious, and naturally contains an amount of inherent risk. Namely, it is unclear whether or not a complete family of higher-order monogamy relations or weight enumerators exists, and how it can be found analytically. To mitigate these risks, we approach the research questions from two directions: First, we will analyze small systems, where we are confident that an analytic analysis can be done. Second, these relations (i.e. a Hermitian idempotent in S_k^n) can also be obtained with the help of a computer algebra system such as Mathematica or Maple. Thus Milestones 1.1, 1.2, 2.1 and parts of Milestone 1.3 are thus certainly feasible, even if analytical methods fail.

The supervisor Prof. Jens Siewert has a strong expertise on monogamy relations and invariants [30, 35, 36, 40, 41], in particular being an author of Ref. [13] that introduced a degree-four monogamy relation. Together with the knowledge that I acquired during my PhD on multipartite entanglement and on the quantum weight enumerator theory, we are confident that we can successfully address the challenges of this ambitious project.

2.4. Schedule and milestones

Month 1-6: Focus on Milestones M 1.1 and M 1.2. To start, we analyze the higher-order shadow inequalities in the context of existing monogamy relations and invariants. By the end of the first six months, we aim to obtain an explicit higher-order relation for three or four parties.

Month 7-12: Focus on Milestones M 1.3 M 2.1. During the next six months, we will derive corresponding compatibility conditions and positive maps for entanglement detection. By the end of the first year, we aim to have the algorithm to find Hermitian idempotents implemented. We will provide the algorithm as well as tables of idempotents online repositories such as github.

Month 13-18: Focus on Milestones M 2.2 and 2.3. In the last six months, we will focus on obtaining families of higher-order shadow inequalities. We expect that these indeed will be interesting monogamy relations, which can be cast into corresponding weight enumerators. With these, we will aim to solve open cases of putative QECC and AME states and adress the QMDS conjecture.

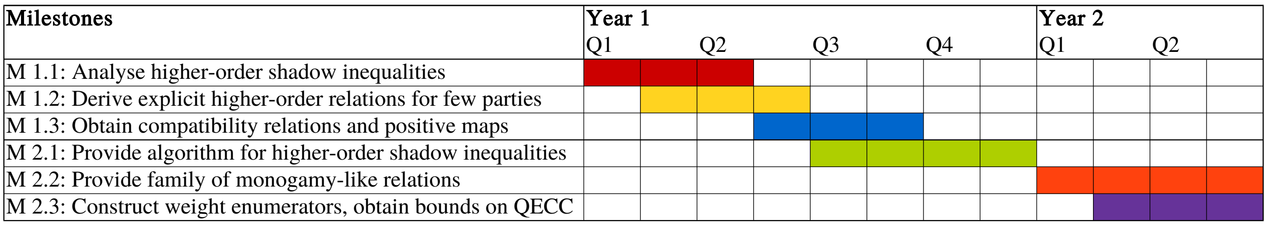

FIG. 1. Schematic view of the time-table for the project.

2.5. Reason for the choice of the research institution

The University of the Basque Country (UPV/EHU) has dedicated research groups in the fields of quantum information, science and technology under the QUINST umbrella organization. These include groups working on quantum information, quantum optics and cold atoms, quantum control, spintronics, quantum metrology, atom interferometry, superconducting qubits and circuit QED, and on the foundations of quantum mechanics. This represents an active environment for scientific knowledge exchange and for close interdisciplinary collaboration.

In particular, the supervisor for this project, Prof. Jens Siewert, has a remarkable knowledge in the fields of entanglement characterization by means of local invariants and monogamy relations. His expertise will be indispensable for addressing the research challenges presented in this proposal. We have an active and close collaboration, from which two successful projects resulted in a short time [A], [E]. A third project on monogamy relations of order two, related to the shadow machinery, is currently under its way [F]. I regard an intensified cooperation with him in the form of post-doctoral studies under his supervision and guidance as the next logical step for my academic career.

Furthermore, my current institution, the University of Siegen, and the University of the Basque Country are strongly connected. The groups of Prof. Otfried Gühne, Prof. Geza Tóth, Prof. Jens Siewert, and Dr. Matthias Kleinmann have collaborated extensively together, joint group retreats as well as numerous visits and exchanges of personnel have only strengthened these ties. My transfer to Bilbao thus would foster this close cooperation, boosting the transfer of skills and expertise between the two universities.

2.6. Relevance and impact

The successful conclusion of this project will most likely have substantial impact and relevance: The planned research addresses timely questions in the fields by merging techniques of multipartite entanglement and quantum error-correcting codes. It provides a novel approach to monogamy and compatibility relations for multipartite entanglement and introduces new tools to characterize QECC.

To our knowledge, this proposal is the first systematic approach to obtain complete families of monogamy-like relations for multipartite quantum states in all finite dimensions. Understanding features of multipartite entanglement is not only relevant for quantum information processing tasks, but also for many applications in many-body physics. In particular, understanding how the distribution of quantum correlations is constrained represents a key ingredient for the analysis of condensed matter systems and to numerically calculate ground state energies in theoretical chemistry [42]. Furthermore, by extending the theory of weight enumerators, we anticipate that we will be able to approach many open problems in multipartite entanglement and QECC, such as the QMDS conjecture and stronger bounds on the existence highly entangled states. This will also have direct implications for classical codes, because of the equivalence between quantum stabilizer codes and classical codes over finite fields [32, 43]. By merging methods from the fields of multipartite entanglement and QECC, our work will have a broader impact by bringing these communities together. Thus the expected results are very likely to spur further work.

We will publish our work in internationally visible open-access journals and disseminate the results at international workshops and conferences. The tables of Hermitian idempotents and the program to obtain these will be made available online for the use of other researchers.

2.7. Relevance for personal career development

I regard this early Postdoc.Mobility fellowship as the integral next step in my career path towards becoming a group leader in science. It allows me to carry out my own research program as an independent post-doctoral scientist, and to establish my expertise at the intersection of quantum error-correcting codes and multipartite entanglement.

The fellowship will give me the opportunity to strongly expand and deepen my knowledge and skill set: working in the multidisciplinary QUINST group has the potential to yield new cross-disciplinary collaborations, and will widen my scientific horizon significantly. The supervisor Prof. Jens Siewert has a unique expertise on monogamy relations, and I can learn crucial techniques used for invariants and monotonicity proofs from him. I also look forward to work with his collaborators, such as Christopher Eltschka from the University of Regensburg, Marcus Huber from the IQOQI Institute in Vienna, and others. My expanded professional network will be a key ingredient for my academic career and my subsequent return to Switzerland.

Next to research, I will have ample opportunities to hone my competence in lecturing and supervision. In the context of the University of the Basque Country's Master program on Quantum Science and Technology, I aim to teach block courses on aspects of multipartite entanglement, classical and quantum error-correcting codes, and on stabilizer states. Also, I will co-supervise a PhD student joining the group of Prof. Jens Siewert next year. This will further contribute to my hirability, teaching and supervision being an integral part of academia.

With no doubt, the successful conclusion of this research program will substantially improve my long-term academic employability and career prospects. The skills acquired and collaborations formed will be highly valuable for my return to Switzerland, where I plan to obtain competitive funding such as a Marie Skłodowska-Curie fellowship or an Ambizione grant as my next step towards a permanent position in science.

3. BIBLIOGRAPHY

[1] O. Gühne and G. Tóth, Physics Reports **474**, 1 (2009).
[2] German National Academy of Sciences Leopoldina, *Quantum Technology: From research to application* (2015).
[3] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States, An Introduction to Quantum Entanglement*, Cambridge University Press (2017).
[4] A. J. Scott, Phys. Rev. A **69**, 052330 (2004).
[5] B. Terhal, IBM J. Res. Dev. **48**, 71 (2004).
[6] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[7] E. M. Rains, IEEE Trans. Inf. Theory **45**, 2361 (1999).
[8] E. M. Rains, IEEE Trans. Inf. Theory **46**, 2361 (2000).
[9] W. Hall, Phys. Rev. A **72**, 022311(2005).
[10] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
[11] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004).
[12] H. S. Dhar, A. K. Pal, D. Rakshit, A. S. De, U. Sen, arXiv:1610.01069.
[13] C. Eltschka and J. Siewert, Phys. Rev. Lett. **114**, 140402 (2015).
[14] B. Toner, Proc. R. Soc. A **465**, 59 (2009).
[15] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).
[16] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Phys. Rev. A **92**, 032316 (2015).
[17] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.-K. Lo, Phys. Rev. A **86**, 052335 (2012).
[18] A. Higuchi and A. Sudbery, Phys. Lett. A **273**, 213 (2000).
[19] E. M. Rains, IEEE Trans. Inf. Theory **44**, 1388 (1998).
[20] P. Shor and R. Laflamme, Phys. Rev. Lett. **78**, 1600 (1997).
[21] G. Nebe, E. M. Rains, and N. J. A. Sloane, *Self-Dual Codes and Invariant Theory*, Springer Berlin-Heidelberg (2006).
[22] E. M. Rains, IEEE Trans. Inf. Theory **44**, 134, (1998).
[23] M. Grassl, M. Rötteler, and T. Beth, Phys. Rev. A **58**, 1833 (1998).
[24] A. Sudbery, J. Phys. A: Math. Gen., **34**, 643 (2001).
[25] M. S. Leifer, N. Linden, and A. Winter, Phys. Rev. A **69**, 052304 (2004).
[26] M. S. Williamson, M. Ericsson, M. Johansson, E. Sjöqvist, A. Sudbery, V. Vedral, and W. K. Wootters, Phys. Rev. A **83**, 062308 (2011).
[27] S. Szalay, J. Phys. A: Math. Theor. **45**, 065302 (2012).
[28] J. Biamonte, V. Bergholm, and M. Lanzagorta, J. Phys. A: Math. Theor. **46**, 475301 (2013).
[29] A. Osterloh and J. Siewert, Phys. Rev. A **72**, 012337 (2005).
[30] C. Eltschka and J. Siewert, J. Phys. A: Math. Theor. **47**, 424005 (2014).
[31] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. **89**, 207901 (2002).
[32] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, IEEE Trans. Inf. Theory **44**, 1369 (1998).
[33] U. Fano, Rev. Mod. Phys. **19** 74 (1957).
[34] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206(1999).
[35] C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, Phys. Rev. A **85**, 022301 (2012).
[36] O. Viehmann, C. Eltschka, J. Siewert, Phys. Rev. A **83**, 052330 (2011).
[37] P. Butterley, A. Sudbery, and J. Szulc, Found. Phys., **36**, 83 (2006).
[38] See M. Grassl, *Bounds on the minimum distance of linear codes and quantum codes*, available at `www.codetables.de`.
[39] M. Grassl and M. Rötteler, Proc. IEEE Int. Symp. Inf. Theory, pp. 1104-1108 (2015).
[40] A. Osterloh and J. Siewert, Phys. Rev. A **86**, 042302 (2012).
[41] A. Osterloh and J. Siewert, Int. J. Quantum Inform. **04**, 531 (2006).
[42] National Research Council, *Mathematical Challenges from Theoretical/Computational Chemistry* (2005). Washington, DC: The National Academies Press.
[43] D. Gottesmann, *Stabilizer Codes and Quantum Error Correction*, Ph.D. thesis, quant-ph/9705052.